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NILPOTENCE AND FINITE H-SPACES

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ABSTRACT
Zabrodsky asked when is the iterated commutator map $X^n \to X$ for a connected associative H-space X a null map. In this paper we reduce this question to a cohomological question and answer it in several cases.

Introduction

Let X be a finite connected, associative H-space. The functor $[\ \ , X]$ takes its values in the category of groups. In fact,

PROPOSITION 1.1. *The functor* $[\ \ , X]$ *takes its values in pro-nilpotent groups.*

PROOF. If Y is finite dimensional of dimension d, then any $(d+1)$-fold commutator of maps $Y \to X$ is zero since it factors through $X^{(d+1)}$ which is d-connected.

Any Y can be expressed as

$$Y = \varinjlim Y^\alpha$$

with Y^α finite. In the exact sequence

$$\varprojlim{}^1[\Sigma Y^\alpha, X] \to [Y, X] \to \varprojlim [Y^\alpha, X]$$

the quotient group is visibly pro-nilpotent. It suffices to show that for some n, the intersection

$$\Gamma_n \cap \varprojlim{}^1[\Sigma Y^\alpha, X]$$

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is zero. Now the subgroup $\varprojlim^1[\Sigma Y^\alpha, X]$ consists of phantom maps, so the result follows from:

PROPOSITION 1.2. *Let*

$$f_i : Y_i \to X, \qquad i = 1, \ldots, d$$

be maps from d spaces to a d-dimensional finite H-space. If the left normalized commutator

$$f = [f_1, \ldots, f_d]$$

is a phantom map, it is null.

LEMMA 1.3. *Let T: Spaces → Spaces be a functor preserving directed colimits (up to homotopy), and sending nullhomotopic maps to nullhomotopic maps. If $f: X \to Y$ is a phantom map, then so is $Tf: TX \to TY$.*

PROOF. Phantom maps are the directed colimits of null maps. Writing $f = \varinjlim f^\alpha$, with f^α null, gives a presentation

$$Tf = \varinjlim Tf^\alpha$$

with Tf^α null. This completes the proof.

The group $[Y_1 \times \cdots \times Y_n, X]$ is naturally isomorphic to

$$[Y_1 \wedge \cdots \wedge Y_n, X] \times [T^n(Y_1, \ldots, Y_n), X],$$

where

$$T^n(Y_1, \ldots, Y_n) = \{(y_1, \ldots, y_n) \in Y_1 \times \cdots \times Y_n \mid \text{some } y_i = *\}$$

denotes the fat wedge. For $f \in [Y^n, X]$, let $f_\wedge \in [Y \wedge \cdots \wedge Y, X]$, and $f_T \in [T^n(Y), X]$ denote the components of f.

COROLLARY 1.4. *A map $f: Y_1 \times \cdots \times Y_d \to X$ is phantom if and only if each component*

$$f_\wedge : Y_1 \wedge \cdots \wedge Y_d \to X \qquad \text{and} \qquad f_T : T_n(Y_1, \ldots, Y_d) \to X$$

is phantom.

PROOF. Consider the following diagram:

$$\begin{array}{ccccc} & & Y_1 \times \cdots \times Y_d & \xrightarrow{f} & X \\ & & \downarrow & & \downarrow \\ T_n(Y_1, \ldots, Y_d) \vee (Y_1 \wedge \cdots \wedge Y_d) & \longrightarrow & \Omega\Sigma(Y_1 \times \cdots \times Y_d) & \xrightarrow{\Omega\Sigma f} & \Omega\Sigma X \longrightarrow X \end{array}$$

If f is phantom, so is $\Omega\Sigma f$ by the lemma, hence so are the components of f since they factor through $\Omega\Sigma f$. Conversely, if the components of f are phantom so is

$$\Omega\Sigma(f_T \vee f_\wedge) \sim \Omega\Sigma f$$

by the lemma and hence f since it is a retract of $\Omega\Sigma f$. This completes the proof.

Proof of Proposition 1.2. We may assume that the Y_i are connected. By the corollary, the components of f are phantom. But f_T is null, and

$$f_\wedge \in [Y_1 \wedge \cdots \wedge Y_d, X].$$

There are no non-trivial phantom maps in $[Y_1 \wedge \cdots \wedge Y_d, X]$, since $Y_1 \wedge \cdots \wedge Y_d$ is d-connected, and $\pi_* X$ is finite for $* > d$. This completes the proof of the proposition.

Zabrodsky [Z, §2.6] asked when such a functor takes its values in nilpotent groups.

Lemma 1.5. *For a finite H-space X, the following are equivalent:*
(1) $[\ \ , X]$ *is nilpotent group valued.*
(2) *For $n \gg 0$ the left normalized iterated commutator maps*

$$c_n : X^n \to X$$

are nullhomotopic.

Proof. The implication (2)$\Rightarrow$(1) is trivial, so suppose that $[\ \ , X]$ is nilpotent group valued. Let W be a countably infinite product of copies of X and suppose that $[W, X]$ is nilpotent of class $< n$. Then the left normalized commutator of the first n projections

$$W \to X^n \xrightarrow{c_n} X$$

is null. But the map $W \to X^n$ is projection to a factor, so the iterated commutator map $c_n : X^n \to X$ is null.

DEFINITION (Zabrodsky). A finite connected associative H-space X is h-nilpotent if equivalently:

(1) $[\ \ , X]$ is nilpotent group valued.

(2) For some n, the iterated left normalized commutator map

$$c_n : X^n \to X$$

is null.

For the sake of optical harmony, let's refer to the component $c_{n\wedge}$ of c_n as $\hat{c}_n$. The component c_{nT} is zero.

Zabrodsky also defined the weaker notion of H-solvable and showed that the Lie groups SO(n), $U(n)$, and Sp(n) are H-solvable. I can see no reason not to make the following conjecture:

CONJECTURE. *A connected, homotopy associative finite H-space is h-nilpotent.*

The point of this note is to describe a cohomological criteria for h-nilpotence and to verify the above conjectures in certain cases.

It gives me great pleasure to thank Alex Zabrodsky for bringing this question to my attention and for his interest and encouragement in this work.

§2. Cohomological criteria for H-nilpotence

For each prime p let $K(n)$ denote the nth Morava K-theory at p [R]. Thus

$$K(n)_*(\mathrm{pt}) \approx \mathbf{F}_p[v_n, v_n^{-1}], \qquad |v_n| = 2p^n - 2,$$

and the external cup product

$$K(n)_*X \bigotimes_{K(n)_*} K(n)_*Y \to K(n)_*(X \times Y)$$

is an isomorphism. It follows that when X is a finite H-space, $K(n)_*(X)$ is a finite dimensional Hopf-algebra over $K(n)_*(\mathrm{pt})$.

DEFINITION. A finite rank Hopf-algebra over a ring R is nilpotent if for $n \gg 0$ the iterated left normalized commutator map

$$A^{\otimes n} \xrightarrow{[\ ,\ ]\times 1_{A^{\otimes(n-2)}}} A^{\otimes(n-1)} \longrightarrow \cdots \longrightarrow A$$

is zero.

THEOREM 2.1. *Let X be a finite connected associative H-space. Then X is h-nilpotent if and only if any one of the following conditions is satisfied:*

(1) $MU^*\bar{c}_n = 0$ *for* $n \gg 0$.

(2) *For each prime* p, $BP^*\bar{c}_n = 0$ *for* $n \gg 0$.

(3) $K(n)_*X$ *is a nilpotent Hopf-algebra for all* n, *at all primes* p.

COROLLARY 2.2. *If* $H_*(X; \mathbf{Z})$ *is torsion free, then* X *is h-nilpotent.*

The corollary applies, for example, when $X = U(n)$ or $\mathrm{Sp}(n)$.

PROOF. Note that $H\mathbf{Q}^*c_n = 0$ for $n \gg 0$ since $X^{\wedge n}$ is $(n-1)$-connected. By assumption

$$MU^*(X^{\wedge n}) \rightarrow MU^*(X^{\wedge n}) \otimes \mathbf{Q}$$

is injective, so by the theorem it suffices to show that $MU^*\bar{c}_n \otimes \mathbf{Q} = 0$. But for a finite complex Y, there is a natural isomorphism

$$MU^*(Y) \otimes \mathbf{Q} \approx MU^*(\mathrm{pt}) \otimes H\mathbf{Q}^*Y,$$

so $MU^*\bar{c}_n \otimes \mathbf{Q} = 0$ as soon as $H\mathbf{Q}^*\bar{c}_n = 0$. This completes the proof.

Theorem 2.1 is proved by reducing the problem to one in stable homotopy theory and applying the nilpotence theorem ([DHS]).

§3. Reduction to stable homotopy theory

Let X and Y be pointed finite complexes and $f: Y \wedge X \rightarrow Y$ a map. Define maps

$$\varphi_n : Y \wedge X^{(n)} \rightarrow Y$$

inductively by

$$\varphi_n = \varphi_{n-1} \circ (f \wedge 1),$$

$$Y \wedge X \wedge X^{(n-1)} \xrightarrow{f \wedge 1} Y \wedge X^{(n-1)} \xrightarrow{\varphi_{(n-1)}} Y.$$

PROPOSITION 3.1. *The maps* φ_n *are null for* $n \gg 0$ *if and only if the maps* φ_n *are stably null for* $n \gg 0$.

LEMMA 3.2. *Suppose that* φ_n *is stably null for some* n. *Then there is an integer* M *such that for all finite dimensional* $(M-1)$*-connected* Z, *the maps*

$$\varphi_m \wedge 1_Z : Y \wedge X^{(m)} \wedge Z \rightarrow Y \wedge Z$$

are null for $m \gg 0$.

PROOF. Since φ_n is stably null, and X and Y are finite, there exists an $M > 0$ with

$$S^M \wedge Y \wedge X^{(n)} \to S^M \wedge Y$$

null. Take this value of M. The lemma is now proved by induction on

$$\operatorname{span}(Z) = \dim(Z) - \operatorname{connectivity}(Z),$$

the case $\operatorname{span}(Z) = 1$ having been done above. Let $Z_0 \subset Z$ be any $(M-1)$-connected subspace with the property that

$$\max\{\operatorname{span}(Z_0), \operatorname{span}(Z/Z_0)\} < \operatorname{span}(Z)$$

(for example, take Z_0 to be the $(\dim(Z)-1)$-skeleton of Z). Consider the following enormous diagram:

$$\begin{array}{ccccc}
 & & Z_0 \wedge X^{(n)} \wedge Y & \longrightarrow & Z_0 \wedge Y \\
 & & \downarrow & & \downarrow \\
Z \wedge X^{(2n)} \wedge Y & \xrightarrow{1 \wedge \varphi_n} & Z \wedge X^{(n)} \wedge Y & \longrightarrow & Z \wedge Y \\
\downarrow & & \downarrow & \nearrow & \\
Z/Z_0 \wedge X^{(2n)} \wedge Y & \xrightarrow{1 \wedge \varphi_n} & Z/Z_0 \wedge X^{(n)} \wedge Y & &
\end{array}$$

By the induction hypothesis, the top horizontal arrow is null for $n \gg 0$ so the factorization through the dotted arrow exists. Also, $1_{Z/Z_0} \wedge \varphi_n$ is null for $n \gg 0$ so the bottom left map is null for $n \gg 0$. It follows that the horizontal composition, $1_Z \wedge \varphi_{2n}$, is null for $n \gg 0$. This completes the proof.

PROOF OF PROPOSITION 3.1. Since a null map is stably null, one direction is obvious. Suppose that the maps φ_n are stably null for $n \gg 0$. Let M be as in Lemma 3.2, and take $Z = X^{(M)}$. By the lemma, the first map in the following factorization of φ_{M+m} is null:

$$X^{(M)} \wedge X^{(m)} \wedge Y \to X^{(M)} \wedge Y \to Y.$$

This completes the proof.

COROLLARY 3.3. *Let X be a connected finite H-space. The maps c_n are null for $n \gg 0$ if and only if the maps $\bar{c}_n$ are stably null for $n \gg 0$.*

Call a map $f: X \to Y$ of pointed spaces *smash nilpotent* if

$$f^{(n)}: X^{(n)} \to Y^{(n)}$$

is null for $n \gg 0$. An argument similar to the proof of Proposition 3.1 proves:

PROPOSITION 3.4. *Let $f: X \to Y$ be a map of finite spaces. Then f is smash nilpotent if and only f is stably smash nilpotent.*

§4. Cohomological criteria

Now that the problem of nilpotence has been reduced to a problem in stable homotopy theory, the cohomological criteria of Theorem 1 can be deduced from the nilpotence theorem ([DHS], [HS]).

THEOREM 4.1 ([DHS], [HS]).

(1) *Let*

$$\cdots \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} X_{n+1} \longrightarrow \cdots$$

be a sequence of connected spectra, with X_n c_n-connected, and with $MU_* f_n = 0$ for all n. If

$$c_n \geqq an + b$$

for some a, b, then $\varinjlim X_n$ is contractible.

(2) *A map $f: X \to Y$ of finite spectra is smash nilpotent if for all $0 \leqq n \leqq \infty$, and for all primes p, $K(n)_* f = 0$.*

Let Y be the Spanier–Whitehead dual of the suspension spectrum of X, and let $\varphi_n: Y \to Y^{(n)}$ be the dual of $\bar{c}_n$. Let T be the homotopy colimit of the sequence

$$\text{(S)} \qquad Y \longrightarrow Y^{(2)} \longrightarrow \cdots \longrightarrow Y^{(n+2)} \xrightarrow{\varphi_2 \wedge 1} Y^{(n+3)} \longrightarrow \cdots.$$

LEMMA 4.2. *For a ring spectrum E, the following are equivalent:*

(1) *There is an n with $E_* \varphi_n \wedge 1_{Y^{(m)}} = 0$ for all m.*

(2) *Given m there is an n with $E_* \varphi_n \wedge 1_{Y^{(m)}} = 0$.*

(3) *$E \wedge T$ is contractible.*

(4) *There is an n for which $Y \to Y^{(n)} \to E \wedge Y^{(n)}$ is null.*

(5) *There is an n for which $E \wedge Y \to Y^{(n)} \to E \wedge Y^{(n)}$ is null.*

PROOF. The only non-trivial implications are (3)⇒(4) and (4)⇒(5). Assume (3). Since Y is finite, the nullhomotopy of

$$Y \to E \wedge T = E \wedge \varinjlim Y^{(n)}$$

occurs at some finite value of n. This is assertion (4). Given (4), the first map in the factorization

$$E \wedge Y \longrightarrow E \wedge E \wedge Y^{(n)} \xrightarrow{\mu \wedge 1} E \wedge Y^{(n)}$$

of $E \wedge Y \to E \wedge Y^{(n)}$ is null. This gives (5) and completes the proof of Lemma 4.2.

PROOF OF THEOREM 2.1(1). Take $E = MU$ in Lemma 4.2. By assumption, part (2) is satisfied. By (1), the sequence

$$Y \xrightarrow{\varphi_n} Y^{(n)} \xrightarrow{\varphi_n \wedge 1_{Y^{(n-1)}}} Y^{(2n-1)} \longrightarrow \cdots,$$

obtained by refining (S), satisfies condition (1) of Theorem 4.1. It follows that T is contractible. Now take $E = S^0$ and use the implication (3)$\Rightarrow$(5) of Lemma 4.2 to complete the proof.

PROOF OF THEOREM 2.1(2). By assumption $BP \wedge Y$ is contractible for all p. It follows that $MU \wedge T$ is contractible. Taking $E = MU$ in Lemma 4.2 and using (3)$\Rightarrow$(2) reduces to part (1).

PROOF OF THEOREM 2.1(3). This part is similar to the proof of part (2) of Theorem 4.1 (see [HS]). We need some auxiliary spectra. For more information see [R] or [HS]. Let p be a fixed prime and let $P(n)$ be the quotient of the spectrum BP by the ideal $I_n = (p, v_1, \ldots, v_{n-1})$. Thus

$$P(0) = BP,$$

$$P(n)^*(\mathrm{pt}) \approx \mathbf{F}_p[v_n, v_{n+1}, \ldots],$$

there are natural maps

$$P(n) \to P(n+1)$$

and the limit

$$\lim_{n \to \infty} P(n) = \mathbf{HF}_p$$

is the Eilenberg–MacLane spectrum. The Bousfield classes $\langle BP \rangle$, $\langle K(m) \rangle$, and $\langle P(n) \rangle$ are related by

(B) $$\langle BP \rangle = \langle K(0) \vee \cdots \vee K(n-1) \rangle \vee \langle P(n) \rangle.$$

By assumption, $K(m) \wedge T$ is contractible for all $0 \leqq n \leqq \infty$. Showing that

$BP \wedge T$ is contractible will reduce to part (2). In view of the Bousfield equivalence (B) it suffices to show that $P(n) \wedge T$ is contractible for $n \gg 0$. By assumption

$$\lim_{n \to \infty} P(n) \wedge T = \mathbf{HF}_p \wedge T$$

is contractible. Since Y is finite, it follows that

$$Y \to T \to P(n) \wedge T$$

is null for $n \gg 0$. Taking $E = P(n)$ in Lemma 3.2 and using the implication $(4) \Rightarrow (3)$ completes the proof.

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